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Nonlinear Gauge Theory of Poincaré Gravity

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A Poincaré affine frame bundle $\hat{P}(M)$ and its associated bundle $\hat{E}$ are established. Using the connection theory of fiber bundles, nonlinear connections on the bundle $\hat{E}$ are introduced as nonlinear gauge fields. An action and two sets of gauge field equations are presented.

1. FIBER BUNDLE DESCRIPTION

In this paper the global Poincaré invariance of space-time is extended to a local Poincaré invariance; and the space-time obtained is denoted by M. It is known that the proper Poincaré group $ISO(3, 1)$ is the semidirect product $ISO(3, 1) = T \circledS SO(3, 1)$ of the translation group T and the proper Lorentz group $SO(3, 1)$, and

$$\frac{ISO(3, 1)}{SO(3, 1)} = M' \quad \text{(Minkowski space)}$$

Moreover the Lie algebra $iso(3, 1)$ is the semidirect sum $iso(3, 1) = t \oplus so(3, 1)$ of the Lie algebras t and $so(3, 1)$. We know that $\forall g \in ISO(3, 1)$, g may be written in the form $g = e^{\xi P} e^{HI}$ where $\xi P = \xi^i P_i$, $e^{\xi P} \in T$, $HI = H^{ij} I_{ij}$, $e^{HI} \in SO(3, 1)$, the P_i are generators of the group T, and the I_{ij} are generators of the group $SO(3, 1)$. The Latin indices i, j, k take the values 0, 1, 2, 3, by convention. It may be considered that the group T is identical to its Lie algebra t, and is a Minkowski space:

$$T = t = M'$$

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We also have the mappings

$$e^{\xi} \underset{\ln}{\overset{\exp}{\rightleftarrows}} \xi \tag{1}$$

and

$$\tilde{T} \underset{\ln}{\overset{\exp}{\rightleftarrows}} M' \tag{2}$$

Here $\xi \in M'$, $e^{\xi} \in \tilde{T}$, and $\tilde{T}$ is the multiplicative group obtained from M' through the exponential mapping. Since $\forall g \in ISO(3, 1)$, $g = e^{\xi P} e^{HI}$ is valid, here (ξ^i, H^{ij}) are canonical coordinates of g, i.e., $\forall g = e^{\xi P} e^{HI} \in ISO(3, 1)$, we have $e^{\xi P} e^{HI} \to (\xi^i, H^{ij}) \in iso(3, 1)$. Similarly, $\forall e^{\xi P} \in T$, its canonical coordinates in M' are ξ^i.

Now we begin to construct an exponent bundle $\tilde{E} = \tilde{E}(M, \tilde{T}, ISO(3, 1))$. Let $g = e^{\xi P} e^{HI}$ be an arbitrary element of $ISO(3, 1)$, and let $r = e^{\eta} P (\leftrightarrow e^{\eta} \in \tilde{T})$ be an arbitrary element of T. Then $ISO(3, 1)$ acts on $\tilde{T}$ from the left by

$$gr = e^{\xi P} e^{HI} e^{\eta P} = e^{\eta' P} e^{HI} \tag{3}$$

where

$$e^{\eta P} \overset{g}{\mapsto} e^{\eta' P}, \qquad \eta' = \eta'(\eta, g) \tag{4}$$

is a realization of $ISO(3, 1)$ on $\tilde{T}$, and

$$\eta' = \eta'(\eta, g) = \xi + \mathrm{Ad}(e^{HI})\eta$$

is a mapping determined by the combination law of group $ISO(3, 1)$. Here $\mathrm{Ad}(e^{HI})$ is an adjoint representation of $SO(3, 1)$. As the structure group $ISO(3, 1)$ is localized, $\forall X \in M$, the set of exponent group elements that take values at point X, is denoted by $\tilde{T}_x$. Then $\tilde{T}_x$ is a fiber over point X. It is easy to see that $\tilde{T}_x$ is isomorphic to $\tilde{T}$; we denote the isomorphism by $\tilde{T}_x \sim \tilde{T}$. Taking the union $\tilde{E} = \bigcup_{x \in M} \tilde{T}_x$ of $\tilde{T}_x$ at all $X \in M$, then we obtain the M-base, $\tilde{T}$-fiber, $ISO(3, 1)$-structure group exponent bundle $\tilde{E} = \tilde{E}(M, \tilde{T}, ISO(3, 1))$. If e^{η} is an element of $\tilde{T}$, then every element $e^{\zeta} \in \tilde{T}_x$ will give a mapping $\tilde{u}: \tilde{T} \to \tilde{T}_x$, $e^{\eta} \mapsto e^{\zeta}$. If we denote $\tilde{u}$ by e^{ζ} in $\tilde{T}_x$, then the action of the group $ISO(3, 1)$ on $\tilde{E}$ can be defined as $g\tilde{u} = \tilde{u}(ge^{\eta P}) = e^{\bar{\zeta} P} e^{HI} \leftrightarrow e^{\bar{\zeta}}$, where $g = e^{\xi P} e^{HI} \in ISO(3, 1)$, $ge^{\eta} = e^{\eta' P} e^{HI} \leftrightarrow e^{\eta'} \in \tilde{T}$, and $\bar{\zeta} = \zeta + \eta'$. We have, $\forall X \in M$, a bundle projection π of $\tilde{E}$ on M, which maps point $\tilde{u} = e^{\zeta} \in \tilde{T}_x$ onto point X, i.e., $\pi(\tilde{u}) = X$. The cross section of bundle $\tilde{E}$ will be a differential distribution of the exponent group element, and may be obtained by the exponent mapping from the translation group $T(X) = M'(X)$. We have $\tilde{E} \sim M \otimes \tilde{T}$, and $\dim \tilde{E} = \dim M + \dim \tilde{T} = 4 + 4 = 8$.

We can also construct an associated bundle $\hat{E} = \hat{E}(M, M', ISO(3, 1))$ corresponding to $\tilde{E}$, for which M is base, the tangent Minkowski space M'

of M is fiber, $ISO(3, 1)$ is structure group, and a principal bundle $\hat{P}(M) = \hat{P}(M, ISO(3, 1))$ associated by $\hat{E}$ may be established also. Due to (1) and (2), some one-to-one correspondences between fiber $\tilde{T}$ and fiber M' can be established. The mapping

$$\eta \overset{g}{\mapsto} \eta', \qquad \eta' = \eta'(\eta, g) = \xi + \mathrm{Ad}(e^{HI})\eta \tag{5}$$

is a realization of group $ISO(3, 1)$ on M'. If $X \in M$, the fiber M'_x over the point X may be obtained from $\tilde{T}_x$ by the mapping $\tilde{T}_x \overset{\ln}{\to} M'_x$. Thus, the mapping maps $e^{\eta} \in \tilde{T}_x$ onto $\eta \in M'_x$. The set thus obtained is a fiber M'_x over X. Taking the union $\hat{E} = \bigcup_{x \in M} M'_x$ at all X of M, we may construct a vector bundle $\hat{E} = \hat{E}(M, M', ISO(3, 1))$.

Of course, the bundle $\hat{E}$ may be used to describe the gauge theory of gravitation (Changgui and Bangqing, 1986), and thus to construct a nonlinear Poincaré gauge gravity (PG). Relation (5) is a transformation of the vector under a transformation of the local affine frame $\{\rho, e_i\}$. Since the group $ISO(3, 1)$ is free, we can find a local affine frame transformation associated with (5) in M':

$$\{\rho, e_i\} \overset{g}{\mapsto} \{\rho', e'_i\}$$

Here $e_i \overset{g}{\mapsto} e'_i(e_i, g)$ is a Lorentz rotation, and $\rho \overset{g}{\to} \rho' = \rho'(\rho, g)$ is a translation.

Thus, under the free action of the group $ISO(3, 1)$, $\forall X \in M$, $\exists$ a series of local affine frame $\{\lambda, X_i\}_x$ [denoted by $\hat{P}_x(M)$], and there is an isomorphism $\hat{P}_x(M) \sim ISO(3, 1)$. The union $\hat{P}(M) = \bigcup_{x \in M} \hat{P}_x(M)$ of $\hat{P}_x(M)$ at all points X of M will be the M-base, $ISO(3, 1)$-structure group, Poincaré affine frame bundle $\hat{P}(M) = \hat{P}(M, ISO(3, 1))$ obtained. For an element $\eta \in M'$, there is an element in M'_x which defines a mapping $\check{u}\colon M' \to M'_x$, $\eta \mapsto \zeta$, i.e., $\check{u}\eta = \zeta$. At the same time, for a local affine frame $\{\rho, e_i\}$ in M', each local affine frame $\{\lambda, X_i\}$ of M'_x can also give a mapping $\hat{u}\colon M' \to M'_x$ and $\hat{u}\{\rho, e_i\} = \{\lambda, X_i\}$. On the fiber $\pi^{-1}(X)$ of principal bundle $\hat{P}(M)$, let $\mathscr{V} = \{\lambda, X_i\}$ be a point of $\pi^{-1}(X)$; then the right action of $ISO(3, 1)$ on $\hat{P}(M)$ may be defined as $\mathscr{V}g = \mathscr{V}'$, where $g \in ISO(3, 1)$, $\mathscr{V}' = \{\lambda', X'_i\}$. For the associated bundle $\hat{E}$, the left action of $ISO(3, 1)$ on its fiber M', using (3) and (4), may be defined as

$$(g, \eta) \in ISO(3, 1) \otimes M' \mapsto g\eta = \eta' \in M'$$

The right action of the group $ISO(3, 1)$ on the product manifold $\hat{P} \otimes M'$ is given by

$$(\mathscr{V}, \eta) \overset{g}{\mapsto} (\mathscr{V}g, g^{-1}\eta)$$

where $g \in ISO(3, 1)$, $(\mathscr{V}, \eta)$, and $(\mathscr{V}g, g^{-1}\eta) \in \hat{P} \otimes M'$. The quotient space

$\hat{P}\otimes M'/ISO(3,1)$ of $\hat{P}\otimes M'$ under group $ISO(3,1)$ is the vector bundle $\hat{E}$. Now, $\forall\mathscr{V}\in\hat{P}(M)$ and $\eta\in M'$, we use $\mathscr{V}\eta$ to denote the image of natural projection $\hat{P}\otimes M'\to\hat{E}$, $(\mathscr{V},\eta)\mapsto\mathscr{V}\eta$; then there exists a mapping $\hat{P}\otimes M'\to M'$ which induces a projection $\pi_{\hat{E}}$ from $\hat{E}$ onto M. Now, $\forall X\in M$, the set $\pi_{\hat{E}}^{-1}(X)$ is a fiber M'_x over X. Any point $\mathscr{V}(\pi(\mathscr{V})=X)$ in $\hat{P}(M)$ may be considered as an isomorphism from M' into $\pi_{\hat{E}}^{-1}(X)$. And $\forall\eta\in M'$, η determines a mapping from $\hat{P}(M)$ into $\hat{E}$; $\forall X\in M$, the bundle projection $\pi_{\hat{E}}$ from $\hat{E}$ onto M maps the point $\check{u}=\zeta$ onto X, i.e., $\pi_{\hat{E}}(\check{u})=X$. The differential distribution of the translation group $T(X)$ on M is a cross section on the bundle $\hat{E}$. And the local affine frame $\{\lambda(X), X_i(X)\}$ on M is a cross section on $\hat{P}(M)$. In the bundle $\hat{P}(M)$ it gives a submanifold which is diffeomorphic to M. The projection π from $\hat{P}(M)$ onto M maps the point $\{\lambda, X_i\}$ of the fiber $\pi^{-1}(X)$ onto the point X. It is easy to see that $\hat{P}(M)\sim M\otimes ISO(3,1)$ and $\dim\hat{P}(M)=14$. Apparently, the vector bundle $\hat{E}=\hat{E}(M, M', ISO(3,1))$ is a bundle associated with the Poincaré affine frame bundle $\hat{P}(M)=\hat{P}(M, ISO(3,1))$.

It follows from the above that when we extend the global Poincaré invariance of space-time to the local Poincaré invariance, the principal bundle $\hat{P}(M)$ and its associated bundle $\hat{E}$ can be established, and $\hat{P}(M)$, $\hat{E}$ are different from the usual bundle $P(M)$, E (Kobayashi and Nomizu, 1963). The bundle $\hat{P}(M)$ and $\hat{E}$ may be used to describe a nonlinear action mechanism of the gauge group $ISO(3,1)$.

2. NONLINEAR GAUGE FIELDS

The connection on the usual Poincaré bundle $P(M)$ is given (Changgui and Bangqing, 1986) by

$$W^a_\mu J_a=\tfrac{1}{2}B^{ij}_\mu I_{ij}+V^i_\mu T_i \tag{6}$$

where $\{J_a\}=\{T_i, I_{ij}\}$ are translation and Lorentz rotation generators of the group $ISO(3,1)$, and V^i_μ (Lorentz vierbein fields) and B^{ij}_μ are the corresponding gauge potentials of the above generators. Under the transformation of the element $g\in ISO(3,1)$, the transformation formula of the above connection is

$$W'^a_\mu J_a=g(W^a_\mu J_a)g^{-1}+g\partial_\mu g^{-1}$$

Using the connection given by (6), a nonlinear connection A^{ij}_μ and K^i_μ can be defined on the exponent bundle $\tilde{E}$ as (Coleman *et al.*, 1963; Callan *et al.*, 1969)

$$\begin{aligned} G^a_\mu J_a &= \tfrac{1}{2}A^{ij}_\mu I_{ij}+K^i_\mu T_i \\ &= e^{-\xi P}(\partial_\mu+\tfrac{1}{2}B^{ij}_\mu I_{ij}+V^i_\mu T_i)\, e^{\xi P} \end{aligned} \tag{7}$$

G^a_μ can give a connection on the bundle $\hat{E}$, and then may be used as the gauge fields of the PG. Now let $e^{\xi P}=t$, $e^{HI}=h$. Then the transformation of A^{ij}_μ and K^i_μ under $gt=t'h'$ [$g\in ISO(3,1)$] may be written as

$$\begin{aligned} \tfrac{1}{2}A'^{ij}_\mu I_{ij} &= h'(\tfrac{1}{2}A^{ij}_\mu I_{ij})h'^{-1}+h'\partial_\mu h'^{-1} \\ K'^i_\mu &= h'(K^i_\mu T_i)h'^{-1} \end{aligned} \tag{8}$$

From (7) we have

$$A^{ij}_\mu = B^{ij}_\mu \tag{9}$$

$$K^i_\mu = V^i_\mu + \partial_\mu \xi^i + \tfrac{1}{2}A^{jk}_\mu \xi^l C^i_{jk,l} \tag{10}$$

Here $C^i_{jk,l}=\eta_{kl}\delta^i_j-(i\leftrightarrow j)$. From (7) and (8), the transformation formulas of the nonlinear gauge fields A^{ij}_μ, K^i_μ are different from the usual Yang–Mills gauge field (Changgui and Bangqing, 1986) B^{ij}_μ, V^i_μ. Apparently, if $\xi=0$, the values of the connections (9), (10) are the same as those of the connections on the principal bundle $P(M)$. Since the action of the group $ISO(3,1)$ on $\hat{E}$ is arbitrary, it is known that the nonlinear translation connection K^i_μ also determines a nonlinear translation connection on the principal bundle $\hat{P}(M)$. K^i_μ may be considered as nonlinear vierbein fields, and $K^i_\mu \in Gl(4,R)$.

By using nonlinear connections A^{ij}_μ and K^i_μ, one can define the covariant derivative as

$$D_\mu = \partial_\mu - \tfrac{1}{2}A^{ij}_\mu I_{ij}$$

and we may obtain the curvature tensor

$$\begin{aligned} \hat{F}^{ij}_{\mu\nu} &= \partial_\mu A^{ij}_\nu - \partial_\nu A^{ij}_\mu + \tfrac{1}{4}C^{ij}_{kl,mn}A^{kl}_\mu A^{mn}_\nu \\ &= \partial_\mu A^{ij}_\nu + A^i_{\mu k}A^{kj}_\nu - (i\leftrightarrow j) \end{aligned}$$

Here

$$C^{ij}_{kl,mn} = \eta_{kn}\delta^i_l\delta^j_m + \eta_{lm}\delta^i_k\delta^j_n - (i\leftrightarrow j)$$

We can also define another covariant derivative

$$\hat{D}_\mu = \partial_\mu - G^a_\mu J_a = \partial_\mu - \tfrac{1}{2}A^{ij}_\mu I_{ij} - K^i_\mu T_i$$

It is easy to prove that

$$[\hat{D}_\mu, \hat{D}_\nu] = -\hat{F}^a_{\mu\nu}J_a = \tfrac{1}{2}\hat{F}^{ij}_{\mu\nu}I_{ij} - \hat{F}^i_{\mu\nu}T_i$$

Here the curvature tensor is

$$\hat{F}^a_{\mu\nu} = \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + C^a_{bc}G^b_\mu G^c_\nu$$

where $[J_a, J_b] = C^c_{bc} J_c$. And for the components $\hat{F}^{ij}_{\mu\nu}$, $\hat{F}^i_{\mu\nu}$ of $\hat{F}^a_{\mu\nu}$, we have

$$\hat{F}^{ij}_{\mu\nu} = F^{ij}_{\mu\nu} = \partial_\mu B^{ij}_\nu + B^i_{\mu k} B^{kj}_\nu - (\mu \leftrightarrow \nu)$$

$$\begin{aligned}\hat{F}^i_{\mu\nu} &= D_\mu K^i_\nu - D_\nu K^i_\mu \\ &= T^i_{\mu\nu} + \tfrac{1}{2} I_{jk}(B^{jk}_\nu \partial_\mu - B^{jk}_\mu \partial_\nu)\xi^i \\ &\quad + \eta_{lk}[\tfrac{1}{2} I_{mn} B^{mn}_\nu B^{ik}_\mu + B^{ik}_{\nu,\mu} + B^{ik}_\nu \partial_\mu - (\mu \leftrightarrow \nu)]\xi^l\end{aligned}$$

Here $T^i_{\mu\nu} = D_\mu V^i_\nu - D_\nu V^i_\mu = V^i_{\nu|\mu} + B^i_{\mu\nu} - (\mu \leftrightarrow \nu)$ is the torsion tensor of the space-time manifold M under the frame of the usual Poincaré bundle $P(M)$. We call $\hat{F}^i_{\mu\nu}$ the torsion tensor of M under the frame of the nonlinear Poincaré bundle $\hat{P}(M)$. Now we see that if the connections we defined on the bundle $\hat{P}(M)$ or $\hat{E}$ are considered as the gauge potentials, we can establish a nonlinear theory of PG, and the potentials and strengths in the theory are different from those in the usual linear theory.

3. ACTION, GAUGE FIELD EQUATIONS

The curvature scalar of the bundle space $\hat{E}$ is invariant under the gauge group $ISO(3, 1)$, so it may be taken as an action. Let $\hat{Z}_A = \{\hat{D}_\mu, T_i\}$ be a base in $\hat{E}$. Then, using the metric of the space-time manifold $\hat{g}_{\mu\nu} = K^i_\mu K^j_\nu \eta_{ij}$ and the metric η_{ij} of the fiber of $\hat{E}$, we can define a metric of $\hat{E}$ as

$$\hat{G}_{AB} = \langle \hat{Z}_A, \hat{Z}_B \rangle$$

Here $\hat{G}_{\mu\nu} = \langle \hat{D}_\mu, \hat{D}_\nu \rangle = \hat{g}_{\mu\nu}$, $\hat{G}_{ij} = \langle T_i, T_j \rangle = \eta_{ij}$, and $\hat{G}_{\mu i} = \hat{G}_{i\mu} = 0$. The *connection* $\dot{\Gamma}^B_{CA}$ on $\hat{E}$ may be given as

$$\dot{D}\hat{Z}_A = \dot{\Gamma}^B_{CA} \hat{Z}_B$$

and the curvature on $\hat{E}$ is

$$\dot{R}^C_{ABD} = \partial_A \dot{\Gamma}^C_{BD} - \dot{\Gamma}^C_{AE} \dot{\Gamma}^E_{BD} - (A \leftrightarrow B) - C^E_{AB} \dot{\Gamma}^C_{ED}$$

Then the curvature scalar of $\hat{E}$ may be obtained as

$$\dot{R} = \hat{R} + R_{M'} - \tfrac{1}{4} F^2 - \tfrac{1}{4} \hat{F}^2$$

Here $\hat{R}$ is the curvature scalar of M, $R_{M'}$ is the curvature scalar of M' (its value is zero), $-\frac{1}{4}F^2 = -\frac{1}{4}F^i_{\mu\nu j} F^{\mu\nu j}_i$ is the kinetic energy term corresponding to the potential A^{ij}_μ, and $-\frac{1}{4}\hat{F}^2 = -\frac{1}{4}F^i_{\mu\nu} F^{\mu\nu}_i$ is the kinetic energy term corresponding to potential K^i_μ. One may choose $\dot{R}$ as the Lagrangian of the nonlinear gauge theory of gravitation, so the action has the form

$$\dot{S} = \int \left(C\mathscr{L}_m + \hat{R} - \frac{\rho}{4} F^2 - \frac{\rho'}{4} \hat{F}^2 \right) K \, d^4x \tag{11}$$

where $\mathscr{L}_m = \mathscr{L}_m(\psi, \psi_{|\mu})$ is a matter field, $K = \det(K^i_\mu) = (-\hat{g})^{1/2}$, $C = 8\pi k$ (k is Newton's gravitational constant), and ρ, ρ' are two gauge gravitational constants to be determined.

Taking A^{ij}_μ and K^i_μ as dynamic variables, through variation of (11), we can obtain the following two sets of gauge field equations of gravity:

$$\hat{R}^i_\mu - \tfrac{1}{2} K^i_\mu \hat{R} = -C\hat{T}^i_\mu + \rho t^i_\mu + \rho' \tau^i_\mu - (E_\mu{}^{i\nu} - E^{i\nu}{}_\mu + E^\nu{}_\mu{}^i)_{|\nu}$$
$$- 2T_{\nu\lambda}{}^\lambda (E_\mu{}^{i\nu} - E^{i\nu}{}_\mu + E^\nu{}_\mu{}^i) \tag{12}$$

$$CS'^\mu_{ij} + M^\mu_{ij} = -F^{\mu\nu}_{ij\|\nu} \tag{13}$$

Here

$$2\hat{t}^i_\mu = -t_r(F_{\mu\nu} F^{\mu\nu}) K^i + \tfrac{1}{4} t_r(\hat{F}_{\lambda\nu} \hat{F}^{\lambda\nu}) K^i_\mu$$

is the energy-momentum tensor of the gauge potential $A^{ij}_\mu = B^{ij}_\mu$,

$$2\hat{\tau}^i{}_\mu = -\hat{F}^{\lambda\nu}_j \hat{F}^j_{\mu\nu} K^i_\lambda + \tfrac{1}{4} \hat{F}^{\lambda\nu}_j F^i_{\lambda\nu} K^i_\mu$$

is the energy-momentum tensor of the gauge potential K^i_μ,

$$S^\mu_{ij} = \frac{1}{K} \frac{\partial(\mathscr{L}_m K)}{\partial A^{ij}_\mu}$$

is the spin current of the matter field ψ,

$$M^\mu_{ij} = \varepsilon \hat{F}^\mu_{ji} - \hat{F}^\lambda_{jk} K^k_\lambda K^\mu_i - \hat{F}^\lambda_{ki} K^k_\lambda K^\mu_j \quad (\varepsilon = 1 - \rho')$$

and $\|$ denotes the twofold covariant derivative in the natural and moving Lorentz frame; $\hat{T}^i_\mu$ is the mass tensor in the moving frame, $E^\mu{}_\nu{}^i = T^\mu{}_\nu{}^i + \delta^{\mu i} T_\nu - \delta^i_\nu T^\mu$ is the modified torsion tensor; and $T_\mu = T_{\mu\nu}{}^\nu$.

When the space-time manifold M is a Riemann space (torsion-free), then equations (12) and (13) become

$$\hat{R}^i_\mu(\{\ \}) - \tfrac{1}{2} K^i_\mu \hat{R}(\{\ \}) = -C\hat{T}^i_\mu - \hat{t}^i_\mu(\{\ \}) \tag{14}$$

and

$$CS^\mu_{ij} = -\rho F^{\mu\nu}_{ij\|\nu}(\{\ \}) \tag{15}$$

If in the Riemann space M we ignore the contributions of gauge actions, equation (15) vanishes and equation (14) degenerates into the Einstein equation

$$\hat{R}^i_\mu(\{\ \}) - \tfrac{1}{2}K^i_\mu \hat{R}(\{\ \}) = -C\hat{T}^i_\mu \tag{16}$$

If M is a Riemann–Cartan space and we do not consider the contributions of gauge actions, equations (12) and (13) become

$$\hat{R}^i_\mu - \tfrac{1}{2}K^i_\mu \hat{R} = -C\hat{T}^i_\mu - (E_\mu{}^{iv} - E^{iv}{}_\mu + E^v{}_\mu{}^i)|v^{-2}T_{v\lambda}{}^\lambda (E_\mu{}^{iv} - E^{iv}{}_\mu + E^{vi}_\mu)$$

$$CS^\mu_{ij} + K^\mu_{ij} = 0$$

Here $K^\mu_{ij} = \hat{F}^\mu_{ij} - \hat{F}^\lambda_{jk}K^k_\lambda K_i - \hat{F}^\lambda_{ki}K^k_\lambda K^\mu_j$ (contortion).

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